Synthesis Modulo Recursive Functions

based on a paper of E. Kneuss, V. Kuncak, I. Kuraj, P. Suter

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What we want

Given a specification

```
def split (xs: List): (List, List) =
  choose {(r: (List, List)) =>
    content(xs) == content(r._1) ++ content(r._2)
}
```

... we want a synthesized program as a solution.

What we rely on

- *Leon* **Verifier** which reasons over user-defined recursive functions. Given a formula, the procedure is
  - Complete for counterexamples:
    `size(xs) < 3` $\rightsquigarrow$ *Counterexample* `1::2::3::nil`
  - Complete for proofs for certain classes of formulas:
    `size(1::2::xs) > 1`

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- **Deductive Synthesis Framework**

Structure

Deductive Synthesis Framework

We have seen this before:

Synthesis Problem

$[\![\bar{a}\ \langle\phi\rangle\ \bar{x}]\!]$

- $\bar{a}$ Set of input variables
- ϕ Synthesis predicate
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Synthesis Problem Solution

$[\![\bar{a} \ \langle\phi\rangle \ \bar{x}]\!] \quad \vdash \quad \langle P \mid \overline{T}\rangle$

- P Precondition
- $\overline{T}$ Program term

Deductive Synthesis Framework

... now we add a **Path condition**:

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- Π **Path condition**

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Inference rules

We have already seen lots of inference rules:

$$\textsc{One-point}\ \frac{[\![\bar{a}\ \langle \Pi \triangleright \phi[x_0 \mapsto t] \rangle\ \bar{x}]\!] \vdash \langle P \mid \bar{T} \rangle \qquad x_0 \notin \text{vars}(t)}{[\![\bar{a}\ \langle \Pi \triangleright x_0 = t \land \phi \rangle\ x_0\,, \bar{x}]\!] \vdash \langle P \mid \mathsf{val}\ \bar{x} := \bar{T};\, (t\,, \bar{x}) \rangle}$$

$$\textsc{Ground}\ \frac{\mathcal{M} \models \phi \qquad \text{vars}(\phi) \cap \bar{a} = \emptyset}{[\![\bar{a}\ \langle \Pi \triangleright \phi \rangle\ \bar{x}]\!] \vdash \langle \mathsf{true} \mid \mathcal{M} \rangle}$$

$$\textsc{Case-split}\ \frac{[\![\bar{a}\ \langle \Pi \triangleright \phi_1 \rangle\ \bar{x}]\!] \vdash \langle P_1 \mid \bar{T}_1 \rangle \qquad [\![\bar{a}\ \langle \Pi \land \neg P_1 \triangleright \phi_2 \rangle\ \bar{x}]\!] \vdash \langle P_2 \mid \bar{T}_2 \rangle}{[\![\bar{a}\ \langle \Pi \triangleright \phi_1 \lor \phi_2 \rangle\ \bar{x}]\!] \vdash \langle P_1 \lor P_2 \mid \mathsf{if}(P_1)\ \{\bar{T}_1\}\ \mathsf{else}\ \{\bar{T}_2\} \rangle}$$

...

We will extend our system by **three new** types of **rules**.

1. $\frac{\cdots}{\cdots}$ Recursion Rule
2. $\frac{}{\cdots}$ Symb. Term Expl. Rule
3. $\frac{}{\cdots}$ Condition Abduction Rule

Recursion Rule (1)

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2. We use a generic schema to solve the problem for each case:

```
def rec(a₀, ā) =
  require(Π₂)
  a₀ match {
    case Nil ⇒ T̄₁
    case Cons(h, t) ⇒
      lazy val r̄ = rec(t, ā)
      T̄₂
  }
ensuring (r̄ ⇒ Φ[x̄ → r̄])
```

Recursion Rule (2)

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$$\frac{[\![\bar{a}\ \langle \Pi_2 \triangleright \phi[a_0 \mapsto \mathsf{Nil}]\rangle\ \bar{x}]\!] \vdash \langle \mathsf{true} \mid \bar{T}_1 \rangle}{[\![a_0\,,\bar{a}\ \langle \Pi_1 \triangleright \phi\rangle\ \bar{x}]\!] \vdash \langle P \mid \mathsf{rec}(a_0,\bar{a})\rangle}$$

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- **Assumption**: Our generators will generate a solution eventually.

Discovering programs (1)

Given a synthesis problem $[\![\bar{a}\ \langle\phi\rangle\ \bar{x}]\!]$ and a generator *genX* matching the type of $\bar{x}$...

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3. If a model is found, try to falsify the program (fixing $\bar{b} := \bar{b}_0$)
   (a) If it can be falsified: Discard
   (b) If it cannot be falsified: Success

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2. Rapidly discard many candidate programs
3. Build a new pruned generator, then run the procedure from before

Example

$$[\![a_0, a_1 \;\; \langle true \triangleright x = a_0 * a_1 \rangle \;\; x]\!]$$

Example

$$\frac{[\![a_1 \ \langle true \triangleright x = 0 * a_1 \rangle \ x]\!] \quad [\![x' a'_0 a_1 \ \langle true \wedge x' = a'_0 * a_1 \triangleright x = (1 + a'_0) * a_1 \rangle \ x]\!]}{[\![a_0, a_1 \ \langle true \triangleright x = a_0 * a_1 \rangle \ x]\!]}$$

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Example

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def genX() =
        if (⋆) a₁
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def genX() =
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P:= a_1 + x'

```
a_1 + x' = (1+a_0') * a_1
a_1 + a_0'*a_1 = (1+a_0') * a_1
```

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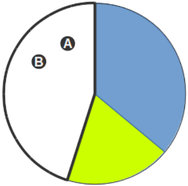
- Typically, a recursive function consists of a top-level case analysis
- **Idea**: Pick program terms, then "abduce" their preconditions until the whole input space is covered

Synthesis Procedure

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if      cond1 then P1
else if cond2 then P2
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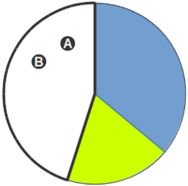


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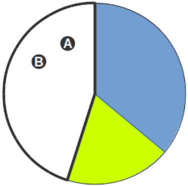


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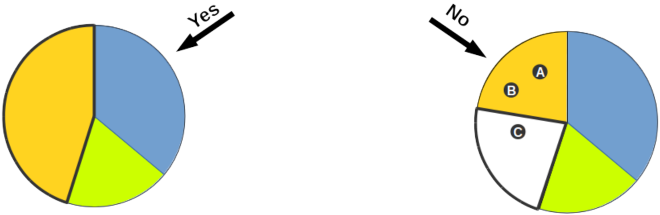

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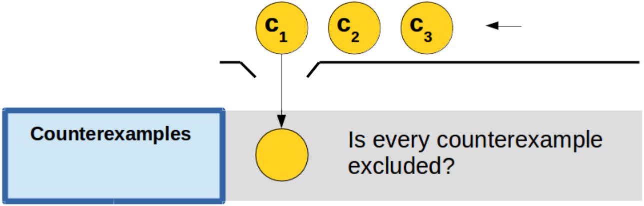

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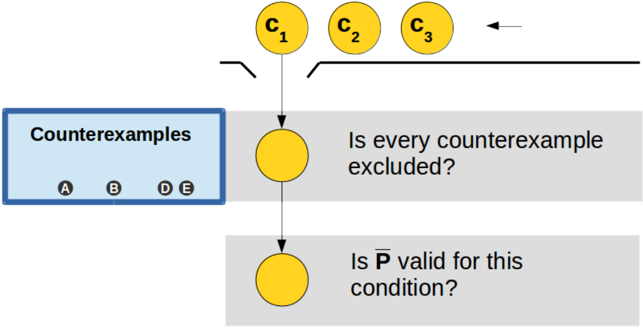

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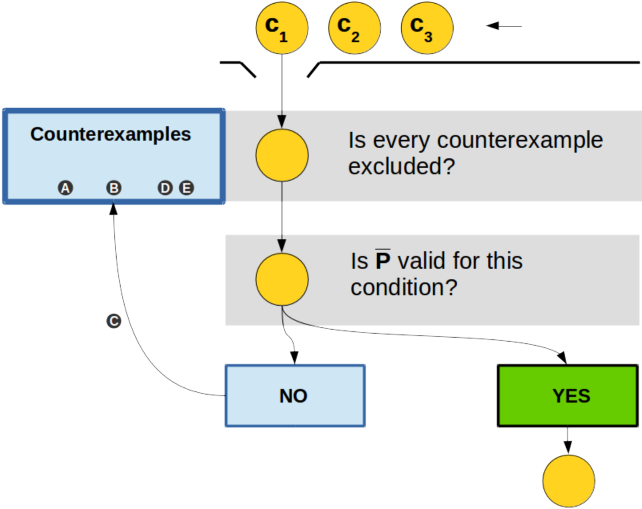

Final assembly

- We have added three new rules which extend the deductive framework
- Synthesis now consists of finding one inference tree for a problem
- At any time, we can stop the synthesis process and use intermediate results

Discussion

- *What are use cases for these techniques, what are their limits?*
- *How can the generator functions best be designed?*
- *How can the search for a synthesis derivation best be organized?*
- *Can ad-hoc interaction during synthesis with the user be considered a strategy?*