Interactive Theorem Proving: An Intro to the Coq Proof Assistant

Presented by Lukas Convent and Prof. Dr. Martin Leucker as part of the *dependable software* course as taught at ISP in Lübeck in 2019.

Learning Goals

- *Programming*: Inductive Data Types and Recursive Functions
- *Specifying*: Encode Logical Formulas as Types
- *Proving*: Prove Logical Formulas about Programs

Outline

Introduction

Functional and Imperative Programs

Definition (Imperative Program)

An **imperative** program p describes a partial function on memory states: Given some initial state σ, the **execution** of p on σ either terminates with a final state σ' or it diverges. For example, the program $\texttt{x} := 2 * \texttt{x}$ maps state $\sigma = \{x \mapsto 1, \_ \mapsto 0\}$ to state $\sigma' = \{x \mapsto 2, \_ \mapsto 0\}$

Definition (Functional Program)

A **functional** program f describes a partial function on values: Given some input value v, the **reduction** of the expression $f(v)$ either terminates in a value v' or it diverges. For example, the program $f(x) := 2 * x$ with 1 given as a value, resulting in the expression $f(1)$, reduces to value 2

Program Verification

- We focus on verifying **functional programs**
- We do not limit ourselves though:
  - Imperative programs can be expressed as functional programs
  - The typical framework to prove properties about imperative programs is the Hoare calculus, which can be easily expressed in out framework
- **Our framework** is a functional language that allows to:
  - Write useful programs
  - Write specifications for these programs
  - Prove these specifications
- **Next:** Recap on what it means to prove a logical statement (such as a specification)

Start simple: Propositional Logic

- Syntax
  - Formulas $\varphi, \psi := p \in AP \mid \bot \mid \varphi \to \psi$
  - Atomic propositions AP
  - (further connectives $\neg, \wedge, \vee, \ldots$ can be used as notation)

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  - Atomic propositions AP
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- Semantics
  - Truth domain $T := \{0, 1\}$
  - Interpretations $v \in AP \to T$
  - Evaluation function

$$\llbracket p \rrbracket_v := v(p)$$

$$\llbracket \bot \rrbracket_v := 0$$

$$\llbracket \varphi \to \psi \rrbracket_v := \begin{cases} 1 & \text{if } \llbracket \varphi \rrbracket_v = 0 \text{ or } \llbracket \psi \rrbracket_v = 1 \\ 0 & \text{otherwise} \end{cases}$$

  - Satisfaction $v \vDash \varphi \quad :\Leftrightarrow \quad \llbracket \varphi \rrbracket_v = 1$
    Validity $\vDash \varphi \quad :\Leftrightarrow \quad v \vDash \varphi$ for all v
  - φ is called a tautology if $\vDash \varphi$

Why *proving*?

- Goal: Given φ, does $\vDash \varphi$ hold?
- First approach: Evaluate and check $[\![\varphi]\!]_v = 1$ for all v

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- Help:
  - Use a proof system
  - Idea: Construct a **finite** proof that φ holds
  - Proof system must be *sound*:
    If φ can be proven ($\vdash \varphi$), then φ is valid ($\models \varphi$)
  - Proof system *may* be *complete*:
    If φ is valid ($\models \varphi$), then φ can be proven ($\vdash \varphi$)

Proof System for Propositional Logic

- Natural deduction via entailment relation $\Gamma \vdash \varphi$
  - Γ is a finite set of formulas $\psi_1, \psi_2, ..., \psi_n$
  - "from Γ, one can **deduce** φ"

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- Defined by inference rules:

$$\varphi \in \Gamma \ \frac{}{\Gamma \vdash \varphi}\ \textsc{Assump} \qquad \frac{\Gamma, (\varphi \to \bot) \vdash \bot}{\Gamma \vdash \varphi}\ \textsc{DoubleNeg}$$

$$\frac{\Gamma, \psi \vdash \varphi}{\Gamma \vdash \psi \to \varphi}\ \textsc{ImpIntro} \qquad \frac{\Gamma \vdash \psi \to \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi}\ \textsc{ImpElim}$$

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- By soundness of $\vdash$, φ is valid

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  1. $3 + 2$
  2. $\textbf{if } true \textbf{ then } true \textbf{ else } false$
  3. $\lambda n.n$
  4. $\lambda n.\ (\lambda b.\ \textbf{if } b \textbf{ then } n \textbf{ else } n + n)$
  5. $\lambda n.\ \textbf{let } sq = n * n \textbf{ in } sq * sq$

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  1. Int
  2. Bool
  3. $\mathit{Int} \to \mathit{Int}$
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  5. $\mathit{Int} \to \mathit{Int}$

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- We write $\vdash t : T$ if term t has type T
- *We can also talk of "soundness" here*: A type system is sound if $\vdash t : T$ implies that t won't "crash" on execution. E.g., `true` $+$ 4 crashes

Proof Trees in Type Systems, 2

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- t itself witnesses its own proof tree of $\vdash t : T$
- *Intuition*: A term itself represents a syntax tree. Put this tree upside down. Traverse the tree, thereby annotating types according to the inference rules. If this works out, you have **the** proof tree. Otherwise, there is none.

Preview: Type System as a Proof System

- You noticed the similarity between the two proof trees?

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- What do we need?
  1. Goal: Find a way of *proving* a specification φ
  2. We encode φ as a type T
  3. We find a term t that is well-typed, i.e. $\vdash t : T$
  4. But this means that t witnesses a proof tree for T
  5. Thus we interpret t as a proof of T and therefore of φ!

Interactive Theorem Provers

- Proofs are **manually written**, potentially with some automatic proof-search aid
- Proofs are **completely formal**
- Proofs can be **automatically checked**
- You have to trust in the soundness of the proof checker
  - Trust is usually established by providing a minimal base of the proof checker

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- Examples: Coq, Isabelle, Agda
- May be based on type theory, but not necessarily
- Applications
  1. Formalized Mathematics, e.g. Four-color theorem in 1976
  2. Correctness Properties
     - Certified C compiler CompCert, started in 2005
     - Soundness of type systems
     - Correctness of protocols
     - Further theorems about formalisms
  3. Generally: Verification where the system model or the property is "too complex" for automatic methods

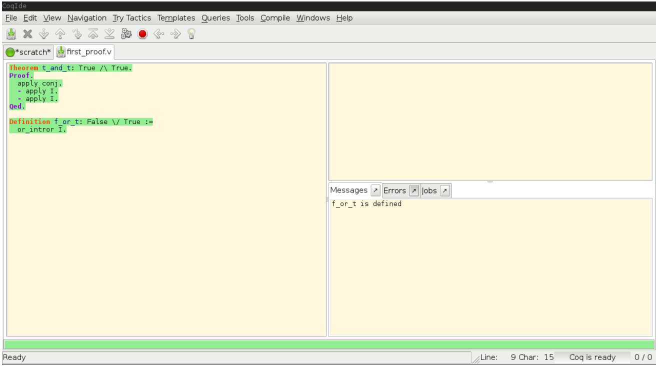
Coq Programming

Coq

- Coq is an interactive theorem prover
- Main idea: Propositions as Types, Proofs as Terms (Curry-Howard-Correspondence)
- One can define
  - **Types (Propositions)**
  - Well-typed **Terms (Proofs)**
- The underlying language Gallina
  - is a dependently-typed functional programming language
  - implements the Calculus of Inductive Constructions
  - is not Turing-complete (every function is total)

Getting started with Coq

1. Installation
   - Win/Mac: Download from `https://coq.inria.fr/`
   - Linux: We recommend installation via OPAM
     `https://coq.inria.fr/opam/www/using.html`
2. IDE
   - Recommendation: Coq IDE, shipped with Coq (see screenshot)
   - Popular plugin for Emacs: Proof General



Coq Programming (Inductive Data Types)

- An inductive data type definition introduces a **new type** and **new well-typed terms**

```
Inductive bool : Type :=
| true  : bool
| false : bool.

Inductive nat : Type :=
| O : nat
| S : nat → nat.
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- `bool`, `nat` are types
- `true`, `false`, `O`, `S` are value constructors

Coq Programming (Definitions)

- A **Definition** gives a name to a term

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Definition two:   nat ≔ S(S O).
Definition three: nat ≔ S(S(S O)).
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- Definitions can be unfolded, which is a kind of *reduction*
- Two terms are *convertible* ($\equiv$) if they reduce to the same term
- E.g., `S two` and `three` are convertible

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  S two
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- *Intuition:* Convertibility is “syntactic equality up-to certain manipulations”

Coq Programming (Functions, Pattern Matching)

- We can define functions that use pattern matching

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Definition negb : bool → bool :=
  fun x ⇒ match x with
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- `fun x ⇒ ...` introduces a function (anonymous, "λ")
- `match ... with | ... end` pattern-matches
- Both constructs introduce a form of reduction and thus of convertibility

```
  negb true
≡ (fun x ⇒ ...) true
≡ match true with | true ⇒ false | ...
≡ false
```

Coq Programming (Short Notation for Functions)

- Recall our function

```
Definition negb : bool → bool :=
  fun x ⇒ match x with
          | true  ⇒ false
          | false ⇒ true
          end.
```

- We can use the following short notation

```
Definition negb (x: bool) : bool :=
  match x with
  | true  ⇒ false
  | false ⇒ true
  end.
```

Coq Programming (Type-Checking)

- In Coq, every term must be well-typed
- What does that mean?
  - We write $\Gamma \vdash t : T$ and call it a *(typing) judgement*
  - "Under context Γ, term t has type T"
  - Context Γ is a list of items $t : T$
  - E.g., we have $x : bool \vdash negb\ x : bool$

    ...but **not** $x : nat \vdash negb\ x : bool$

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- Coq *decides* for a given judgement whether it holds (a.k.a. **type-checking**)

Coq Programming (Type-Checking, Reducing in Coq)

- Type-infer terms and compute (reduce) terms

```
Check (negb true).   ⇝ negb true: bool
Compute (negb true). ⇝ false: bool
```

  - Here, the context Γ is considered by Coq but not explicitly output

Coq Programming (Recursive Functions)

- We can define recursive functions

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Fixpoint plus (m n: nat) : nat :=
  match m with
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  | S m' => S (plus m' n)
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- Above was really a short notation for the following:

```
Definition plus : nat → nat → nat :=
  fix f (m n: nat) :=
    match m with
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    end.
```

Coq Programming (Recursion must be structural)

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- Structural recursion means that recursion is only applied to sub-structures
- Here: `m'` is a sub-structure of `S m'`
- Why this restriction? Remember: Proofs are programs, and non-terminating proofs must be avoided!
  (more later)

Coq Programming (Prelude and Notation)

- Standard data types, functions, notation are pre-defined via the Prelude[1]
- This allows us to write a term like `3 + 2` instead of `plus S(S(S O)) S(S O)`.
- We use the nice notation from now on wherever possible

[1] https://coq.inria.fr/library/Coq.Init.Prelude.html

Coq Programming (Polymorphic Data Types, 1)

- It is often useful to parameterize a data type to avoid multiple definitions such as `natList`, `boolList` etc.

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- Applying this type constructor yields
  - `list nat:  Type`
  - `list bool: Type`
  - ...

Coq Programming (Polymorphic Data Types, 2)

In the parameterized definition

```
Inductive list (X: Type) : Type :=
| nil:  list X
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... the parameter X can be "multiplied-out" to ...

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Inductive list : Type → Type :=
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- The following judgements are introduced
  - `list: Type → Type`
  - `nil: ∀(X: Type), list X`
  - `cons: ∀(X: Type), X → list X → list X`
- The definitions are isomorphic (modulo technicalities), but the parameterized definition emphasises that *the structure* of `list` terms is independ. of the choice of the *"type of content"* `X`

Coq Programming (Implicit Parameters, 1)

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Inductive list (X: Type) : Type :=
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  instead?
- Yes! (See next slide)

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- Recall our example term:
  `cons nat 42 (nil nat) : list nat`

[2] use `Require Import Coq.Lists.List.`

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- This allows to write the example term as:
  `cons 42 nil : list nat`
- But we lack the context to infer `nil : list nat`
- There is further notation[2]: `42::nil : list nat`

[2] use `Require Import Coq.Lists.List.`

Coq Programming (Implicit Parameters, 3)

- Why do we have to bother about **explicit** parameters in expressions?
- In standard programming languages, parameters in terms can be simply deduced by the type checker and are therefore **implicit**, i.e. they are omitted in terms
- Here however, parameters are more subtle and cannot always be deduced
- Take-away: If we are using the usual programming constructs, we just use the options

```
Set Implicit Arguments.
Set Contextual Implicit.
```

and don't have to care about the majority of parameters!

Propositions as Types

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- We need to establish the following mechanisms:
  - A **proposition** (logical statement) is encoded as a **type**
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- We need to establish the following mechanisms:
  - A **proposition** (logical statement) is encoded as a **type**
  - A **proof** of a proposition is encoded as a **term** of that type
- The idea of using propositions as types is also called *Curry-Howard-Correspondence*

The Idea, 2

Spoilers:

Implication $P \to Q$	$\hat{\simeq}$	Funct. type `P -> Q`
Conjunction $P \land Q$	$\hat{\simeq}$	Product type `P × Q`
Disjunction $P \lor Q$	$\hat{\simeq}$	Sum type `P + Q`
Top $\top$	$\hat{\simeq}$	Unit type `1`
Bottom $\bot$	$\hat{\simeq}$	Empty type `0`
Univ. Qu. $\forall(x : A), P(x)$	$\hat{\simeq}$	Π-types
Exis. Qu. $\exists(x : A), P(x)$	$\hat{\simeq}$	Σ-types
Modus Ponens: From $P \to Q$ and P one deduces Q	$\hat{\simeq}$	Function application: `f: P -> Q` and `p: P` gives `f p: Q`
There is a proof tree for P	$\hat{\simeq}$	There is a term `t` such that `t: P`

Top and Bottom

- Let's define the proposition $\top$ ("truth")
- There should be a proof for $\top$

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- There should be **no** proof for $\bot$

```
Inductive False : Prop := .
```

- In Coq, there is a special type universe for propositions, called `Prop` (more about universes later)
- From now on, we write $\top$ for `True` and $\bot$ for `False`

Conjunction

- We now come to our first connective, conjunction

```
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  conj : A → B → and A B.
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- Example: Prove $\top \land \top$

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Definition t_and_t: ⊤ ∧ ⊤ :=
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Inductive or (A B: Prop) : Prop :=
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Definition f_or_t: ⊥ ∨ ⊤ :=
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```

Types, so far

- Let's recall what we know about types in Coq so far
- First, how are types formed? We saw 3 possibilities:

1a) **`Inductive`** types (atomic)

- `bool` : **`Type`**
  - with value `true`, `false`
- $\bot$: **`Prop`**
  - no proofs

1b) **`Inductive`** types (applied type constructors)

- `list nat` : **`Type`**
  - with value `1::2::3::nil`, ...
- $\top \wedge \top$: **`Prop`**
  - with proof `conj I I`

Types, so far

2) Function Types

- `bool → bool` : **Type**
  - with values `negb`, `(fun x ⇒ x)`, ...
- `nat → nat → nat` : **Type**
  - with values `plus`, ...
  - ...
- $\bot \to \bot$: **Prop**
  - with... a proof? Yes: `fun x ⇒ x`
- $\top \to \bot$: **Prop**
  - with... a proof? No.

Types, so far

3) Polymorphic Types

- `∀ (X: Type), list X` : **Type**
  - with one value: `nil`
- `∀ (X: Type), X → list X` : **Type**
  - with value `fun (X: Type) (x: X) ⇒ x::x::x::nil`
  - ...
- `∀ (A B: Prop), A → B → A ∧ B` : **Prop**
  - with proof `conj`
- `∀ (P: Prop), ⊤∨ P` : **Prop**
  - with... a proof? Yes:
    `fun (_: Prop) ⇒ or_introl I`

- **Polymorphic types** are **function types** that take types as arguments
- **Polymorphic values** are **functions** that take types as arguments

Polymorphic Propositions

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- Example: *For all propositions* P, *we have* $\top \vee P$

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- The proof is *polymorphic* in `P`

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Definition true_p:
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- Tactics **generate proof terms**
- Display proof term via `Print t_and_t.`

Proving with Tactics, 2

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- Enables backwards-directed reasoning

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- The *goal* (proof obligation) is simplified/divided/reduced to smaller *subgoals*

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- **`apply`** can be used to apply a value constructor `conj`
- If the value constructor expects further arguments, further subgoals are generated
- This is the case in our example: We have two subgoals $\top$ and $\top$

Tactic: **exact**

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Qed.
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- By **exact**, one can give an explicit proof term

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Proof.
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Qed.
```

- By **exact**, one can give an explicit proof term
- In this example, we give the whole proof term just by a single **exact**, which is equivalent to the two other definitions of t_and_t

Tactic: **intros**

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Lemma p_q_p: ∀ (P Q: Prop), P → Q → P.
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  intros P Q p q.
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- They are now available as **hypotheses** in the context Γ
- Correspondence to proof terms:
  `intros` `x y.` introduces **`fun`** `x y` $\Rightarrow$ ...

Tactic: **destruct** (1)

```
Lemma pq_p: ∀ (P Q: Prop), P ∧ Q → P.
Proof.
  intros P Q H.
  destruct H as [p q].
  apply p.
Qed.
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Tactics: **destruct** (2), **left**, **right**

```
Lemma pq_or_qp:
  ∀ (P Q: Prop), P ∨ Q → Q ∨ P.
Proof.
  intros P Q H.
  destruct H as [p | q].
  - right. apply p.
  - left. apply q.
Qed.
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- For each case, there is a subgoal
- Correspondence to proof terms:
  Introduces **match** ... **with** | ... | ... ⇒ ...
- By **left** (**right**), the first (second) constructor is selected
- Correspondence to proof terms:
  Introduces `or_introl` resp. `or_intror`

Tactic: **split**

```
Lemma and_comm:
  ∀ (P Q: Prop), P ∧ Q → Q ∧ P.
Proof.
  intros P Q H.
  destruct H as [p q].
  split.
  - apply q.
  - apply p.
Qed.
```

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- Correspondence to proof terms:
  Introduces `conj ... ...`

Tactic: **exfalso**

```
Lemma false_proves_anything:
  ∀ (P: Prop), ⊥ → P.
Proof.
  intros P f.
  exfalso.
  exact f.
Qed.
```

- **exfalso** replaces the current goal by $\bot$

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- Crucial part of the corresponding proof term:
  `match` `f` **`with`** *(nothing here)* **`end`**

Tactics: **simpl**, **reflexivity**

```
Lemma negb_tf: negb true = false.
Proof.
  simpl.
  reflexivity.
Qed.
```

- By **simpl**, a goal is maximally reduced

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- How is = encoded as a type and what proof term does **`reflexivity`** introduce?
- The answer is "as an inductive type" but the details are not relevant at this point

Negation

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- You have seen it, as it is already built-in: The negation of a proposition `P` is the implication `P → ⊥`
- We use the notation `~P` for `P → ⊥`
- Example: Prove that for all propositions `P`, we have `P → ~(~P)`.

```
Lemma not_not:
  ∀ (P: Prop), P → ~(~P).
Proof.
  intros P p.
  intros H.
  apply H.
  exact p.
Qed.
```

Types that Depend on Terms

- Recall polymorphic types, i.e. types that depend **on types**

3) Polymorphic Types

- `∀ (X: Type), list X` : **Type**
  - with value `nil`
- $\forall$ `(P: Prop),` $\top \vee$ `P` : **Prop**
  - with proof `fun (_: Prop) ⇒ or_introl I`

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- Types can also depend **on terms**

4) Dependent Types

- `∀ (b: bool), negb (negb b) = b` : **Prop**
- `∀ (n: nat), 0 + n = n` : **Prop**
- `∀ (n: nat), n + 0 = n` : **Prop**
- `∀ (m n: nat), m + n = n + m` : **Prop**

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- `∀ (n: nat), n + 0 = n` : **Prop**
- `∀ (m n: nat), m + n = n + m` : **Prop**

- *Roadmap*: We prove all of the above properties!

Type Universes

- The type of a type is called a *type universe*: Either **`Type`** or **`Prop`**[3]

```
bool : Type    ⊤                    : Prop
nat  : Type    10 = 4               : Prop
               ∀ (P: Prop), ⊤ ∨ P   : Prop
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[3] This is a simplified view that is sufficient for now.

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```
O     : nat            true  : bool
S O   : nat            false : bool
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O      : nat             true  : bool
S O    : nat             false : bool
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```

- A type of **`Prop`** (e.g. ⊤) contains **proofs**

```
I      : ⊤
(fun (_: Type) ⇒ or_introl I)
       : ∀ (P: Prop), ⊤ ∨ P
```

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Back to Booleans, 1

```
Definition negb (x: bool) : bool :=
  match x with
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  end.
```

- Let's prove `negb (negb b) = b` for all `b`

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- The term `negb (neg b)` does not reduce
- Why not? `negb` performs pattern matching, but since we don't know anything about `b` (it could be *any bool*), we don't know which case will match
- But there are only two possible values for `b`. So let's do a case analysis and prove every case!
  1. `b` is `true`. Then `negb (negb true)` reduces to `true`.
  2. `b` is `false`. Then `negb (negb false)` reduces to `false`.

Back to Booleans, 2

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Definition negb (x: bool) : bool :=
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```

- We have all the tools to prove this in Coq

Back to Booleans, 2

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  end.
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```
Lemma negb_inverse:
  ∀ (b: bool), negb (negb b) = b.
Proof.
  intros b.
  destruct b.
  - simpl. reflexivity.
  - simpl. reflexivity.
Qed.
```

Back to Natural Numbers

```
Fixpoint plus (m n: nat) : nat :=
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- We can easily prove that $0 + n = n$ for all n

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Lemma O_plus_n: ∀ (n: nat), 0 + n = n.
Proof.
  intros n.
  simpl.
  reflexivity.
Qed.
```

Natural induction

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- "*any number*" rings a bell: Proof by Natural Induction!
- **Base case.** To show: $0 + 0 = 0$. By definition.
- **Inductive case.** Let IH be $m + 0 = m$. To show: $(S\ m) + 0 = S\ m$. But this is by def. of `plus` convertible to $S(m + 0) = S\ m$. Now use IH, so we have to show $S\ m = S\ m$ which holds by reflexivity of $=$.

Natural induction in Coq, 1

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```
Lemma n_plus_O: ∀ (n: nat), n + 0 = n.
Proof.
  intros n.
  induction n.
  - simpl. reflexivity.
  - simpl. rewrite IHn. reflexivity.
Qed.
```

- **`induction`** `n` does a case-analysis on `n` (like **`destruct`**), but provides an additional inductive hypothesis

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- **`induction`** `n` does a case-analysis on `n` (like **`destruct`**), but provides an additional inductive hypothesis
- **`rewrite`** `IHn` uses the equation `IHn` to substitute a subterm in the current goal

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  `m + n = n + m` for all `m`, `n`

Natural induction in Coq, 2

- Remember the very first property we wanted to show?
  `m + n = n + m` for all `m`, `n`
- Proof by induction over `m`.
  - **Base case**. To show: $0 + n = n + 0$. By our last lemma, we know $n + 0 = n$; by definition of `plus` we know $0 + n = n$; thus we are done.
  - **Inductive case**. Let IH be $m + n = n + m$. To show: $(S\ m) + n = n + (S\ m)$. This goal reduces to $S(m + n) = n + (S\ m)$. By the IH, we can reduce the goal to $S(n + m) = n + (S\ m)$. Here we have the same problem as with proving $n + 0 = n$: It does not hold by definition, because `plus` pattern-matches on the *first* argument. Thus we prove this as an extra lemma, after which the proof is completed.

Natural induction in Coq, 3

- Let's prove the little lemma
  `m + (S n) = S (m + n)` for all `m`, `n`
  ... by induction on `m`

Natural induction in Coq, 3

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  … by induction on `m`

```
Lemma m_plus_S:
  ∀ (m n: nat), m+(S n) = S (m+n).
Proof.
  intros m n.
  induction m.
  - simpl. reflexivity.
  - simpl. rewrite IHm. reflexivity.
Qed.
```

Natural induction in Coq, 4

- Now we can prove commutativity of addition in Coq, following the proof sketch given before

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Lemma plus_comm:
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Proof.
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  induction m.
  - intros. rewrite m_plus_O.
    simpl. reflexivity.
  - intros n. simpl. rewrite IHm.
    rewrite m_plus_S. reflexivity.
Qed.
```

Induction in Coq: Outlook, 1

- We know the principle of natural induction from maths

[4] For further reading: "x is-a-substructure-of y" is well-founded

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- Why have we considered this principle a sound proof method?
- Because our objects (here: natural numbers) are constructed out of **finitely** many steps[4]. We can view a proof of induction as a recipe on how to obtain a proof for **any concrete** number n.

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- Example: How do we prove $3 + 0 = 3$?
  - Use Inductive Case, but need to prove $2 + 0 = 2$. How?
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- Doesn't this proof construction look a lot like... a recursive program?

---

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Induction in Coq: Outlook, 2

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[5] You can take a look e.g. via `Print nat_ind`.

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- In Coq, an **inductive proof** is a **recursive function**
- Every `Inductive` type has this essential property that each object is constructed out of finitely many steps
- There is an induction principle for every type, and it is automatically generated[5]

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- Coq is a **programming language** (with restricted recursion)
- Data is defined inductively, i.e. all values are **finite objects**
- Functions are defined recursively over this inductive structure
- The Curry-Howard-Correspondence provides clever tricks to **encode propositions as types**
- A proposition is proven by a well-typed **proof term**

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- Data is defined inductively, i.e. all values are **finite objects**
- Functions are defined recursively over this inductive structure
- The Curry-Howard-Correspondence provides clever tricks to **encode propositions as types**
- A proposition is proven by a well-typed **proof term**
- A **proof by induction** is a recipe for constructing proofs for **any** element
  - This recipe is a recursive function!

Summary

- Coq is a **programming language** (with restricted recursion)
- Data is defined inductively, i.e. all values are **finite objects**
- Functions are defined recursively over this inductive structure
- The Curry-Howard-Correspondence provides clever tricks to **encode propositions as types**
- A proposition is proven by a well-typed **proof term**
- A **proof by induction** is a recipe for constructing proofs for **any** element
  - This recipe is a recursive function!
- Tactics assist the user in finding a proof term

Literature

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